FIT5124 Advanced Topics in Security

Lecture 1: Lattice-Based Crypto. I

Ron Steinfeld
Clayton School of IT
Monash University

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First Module In a Nutshell

Lattice-Based Cryptography is a cutting-edge cryptographic 'technology'. Has several interesting properties:

- Very fast Public-Key Cryptographic Operations (useful for performance-critical applications).
- Provable Security Guarantees
- Believed 'Post Quantum Computer' Security
- Allows more powerful cryptographic functionalities (in some cases not previously possible), e.g.
  - Fully Homomorphic Encryption (FHE): communication-efficient privacy-preserving computation protocols (later in unit!)

This Lecture: Brief introduction to lattices, hard computational problems, and some related mathematics (more to be introduced gradually in following lectures).

Lecture Outline

Lecture Outline: Motivation and Intro. to Lattice-Based Cryptography

- Lattice-Based Crypto: Brief History
- Lattices: Concepts and intro. to the mathematics
- Lattices: Hard Computational Problems – SVP
- Random Crypto. Lattices: SIS Problem
- SIS Application: Collision-Resistant Hash Function

Following Lectures:

- Cryptanalysis: How Secure is lattice-based crypto? How to choose parameters?
- How to use Lattice-based crypto to build encryption and signature schemes?
- How to make lattice-based crypto. efficient?

Motivation: Why study Lattice-Based Crypto?

Lattice-Based Cryptography has several interesting properties:

- Computational Efficiency: High-speed crypto algorithms
- Novel and Powerful Cryptographic Functionalities (e.g. Fully Homomorphic Encryption – FHE)
- Strong Provable Security Guarantees
- Believed Post Quantum Security

Motivation: Post Quantum World

Today:

- Public-key crypto is essential for secure web transactions.
- Deployed public-key cryptosystems based on Factorization or Discrete-Logarithm problems.

But:

- Shor (1994) showed Fact/DL solvable efficiently on large scale **quantum computer**.
- Quantum computer technology is currently primitive ($15 = 3 \times 5$), but for how long?

Lattice-based crypto seems to resist quantum attacks!

Motivation: Efficiency

Popular cryptosystems are relatively inefficient;
For security level 2^n:

- RSA – key length $\widetilde{O}(n^3)$, computation $\widetilde{O}(n^6)$.
- ECC – key length $\widetilde{O}(n)$, computation $\widetilde{O}(n^2)$.

Structured ('Ring based') Lattices – key length and computation $\widetilde{O}(n)$ **asymptotically, as** n **grows towards infinity**.

In Practice, for typical security parameter $n \approx 100$, with best current schemes, typically have:

- Structured Lattice crypto. **Computation** ≈ 100 times faster than RSA
- Structured Lattice crypto. **ciphertext/key length** $\approx$ RSA key/ciphertext length

Motivation: Provable Security Guarantees

Brief History of Lattice-Based Crypto

- 1978: Knapsack public-key cryptosystem (Merkle-Hellman).
  - Trapdoor One-way Function: $f(x_1, \ldots, x_n) = \sum_{i \leq n} g_i \cdot x_i$.
  - Public: persumably hard knapsack set $(g_1, \ldots, g_n)$.
  - Secret Trapdoor: easy knapsack $(g'_1, \ldots, g'_n)$, $g'_i > 2 \cdot g'_{i-1}$.
  - Public-Secret Relation: $g_i = a \cdot g'_i \bmod q$, $i = 1, \ldots, n$.
- 1982: Poly-time secret recovery attack (Shamir).
- 1980s:

```
for(i = 1; i < N; i++) {
  repair;
  attack;
}
```

Problem with Heuristic Designs: Special random instances – shortcut attacks can exist!

Motivation: Provable Security Guarantees

- 1996: One-Way Func./Encryption with worst case to average case security proof (Ajtai/Ajtai-Dwork) – Introduction of SIS problem.
  - Proof that no shortcut attacks exist – any attack implies solving hard worst-case instances of lattice problems!
- 1996: Efficient ($\widetilde{O}(n)$ time/space) and Practical but heuristic security NTRU encryption (Hoffstein et al) – ideal lattices.
- 2002: Efficient lattice-based one-way function with security proof – ideal lattices (Micciancio).
- 2005: Lattice-Based public-key encryption with security proof – Introduction of LWE Problem (Regev).
- 2005-2015: Many Developments, e.g.
  - Improved Techniques/Proofs (Fourier analysis, Gaussians), Crypto. Hash Functions, Trapdoor signatures, ID-Based Encryption (IBE), Attribute-Based Encryption (ABE), Zero-Knowledge Proofs, Oblivious Transfer, Fully-Homomorphic Encryption (FHE), Cryptographic Multilinear Maps, Program Obfuscation,...

Lattices: Basic Concepts

Point lattices: an area of math. combinining matrix/vector algebra (linear algebra) and integer variables. Both geometry ad algebra play a role.

Before we begin: **Notations**

$\mathbb{Z}$: Set of integers, : $\mathbb{R}$: Set of real numbers $\mathbb{Z}_q$: Ring of integers modulo q

vectors – by default columns: $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, with coordinates b_i, $i = 1, \ldots, n$. Convert to a row vector using transpose: $\vec{b}^T = [b_1 b_2 \cdots b_n]$.

Measures of length (aka norm) for vectors:

- Euclidean norm (aka 'length', '2-norm'): $\|\vec{b}\| = \sqrt{\sum_{i=1}^{n} b_i^2}$.
- Infinity norm (aka 'max' norm): $\|\vec{b}\|_\infty = \max_i |b_i|$.

Lattices: Basic Concepts

Definition

An n-dimensional (full-rank) lattice $L(B)$ is the set of all integer linear combinations of some basis set of linearly independent vectors $\vec{b}_1, \ldots, \vec{b}_n \in \mathbb{R}^n$:

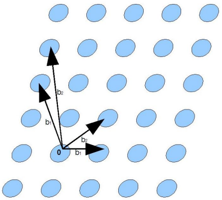
$$L(B) = \{c_1 \cdot \vec{b}_1 + c_2 \cdot \vec{b}_2 + \cdots + c_n \cdot \vec{b}_n : c_i \in \mathbb{Z}, i = 1, \ldots, n\}.$$

- Call $n \times n$ matrix $B = (\vec{b}_1, \ldots, \vec{b}_n)$ a basis for $L(B)$.

Example in 2 Dimensions ($n = 2$)

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix},$$

$$\vec{b}'_1 = \begin{bmatrix} -0.6 \\ 2 \end{bmatrix}, \vec{b}'_2 = \begin{bmatrix} -0.4 \\ 3 \end{bmatrix}$$



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- Call $n \times n$ matrix $B = (\vec{b}_1, \ldots, \vec{b}_n)$ a basis for $L(B)$.
- L is discrete group in $\mathbb{R}^n$, under addition.

Example in 2 Dimensions ($n = 2$)

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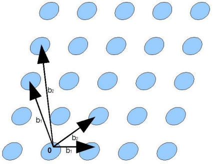


Lattices: Basic Concepts

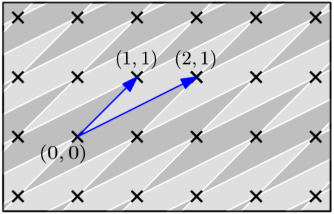
Definition

For an n-dim. lattice basis $B = (\vec{b}_1, \ldots, \vec{b}_n) \in \mathbb{R}^{n \times n}$, the fundamental paralellepiped (FP) of B, denoted $P(B)$, is the set of all real-valued $[0, 1)$-linear combinations of some basis set of linearly independent vectors $\vec{b}_1, \ldots, \vec{b}_n \in \mathbb{R}^n$:

$$P(B) = \{c_1 \cdot \vec{b}_1 + c_2 \cdot \vec{b}_2 + \cdots + c_n \cdot \vec{b}_n : 0 \leq c_i < 1, i = 1, \ldots, n\}.$$

- The translated FPs (in grey in example below) tile the whole n-dim. real vector space $\mathrm{span}(B) = \mathbb{R}^n$ spanned by B.

Example in 2 Dimensions ($n = 2$)



Lattices: Basic Concepts

- There are (infinitely!) many different bases for a lattice.
- Question: Given a lattice L with basis B, how can we tell if B' is another basis for L?
- Geometric Ans.: count L points contained in $P(B')$

Lemma

There is exactly one L point contained in $P(B')$ (the $\vec{0}$ vector) if and only if B' is a basis of L.

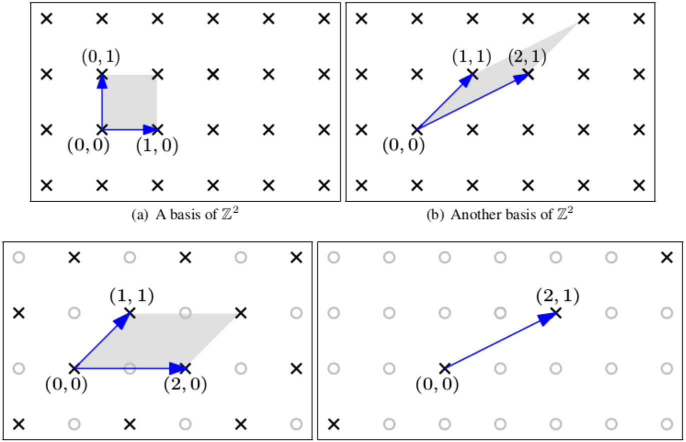
- Algebraic Ans.: Look at determinant of the matrix relating B' to B

Lemma

B' is a basis of $L(B)$ if and only if $B' = B \cdot U$ for some $n \times n$ integer matrix U with $\det(U) = \pm 1$ (we call such a U a unimodular matrix).

Lattices: Basic Concepts

Multiple Bases / FP Examples in 2 dim.



(a) A basis of $\mathbb{Z}^2$

(b) Another basis of $\mathbb{Z}^2$

(c) Not a basis of $\mathbb{Z}^2$

(d) Not a full-rank lattice

Lattices: Basic Concepts

Definition

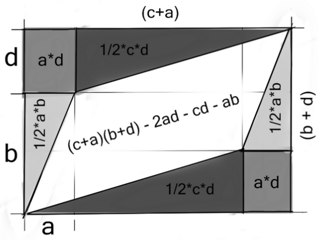
For an n-dim. lattice $L(B)$, the determinant of $L(B)$, denoted $\det L(B)$ is the n-dim. volume of the FP $P(B)$.

Lemma (Equivalent algebraic def. of lattice determinant)

For an n-dim. lattice $L(B)$, we have $\det(L(B)) = |\det(B)|$.

Example of algebraic-geometric relation in 2-dim.:

$$B = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$



- Consequence: For a large n-dim ball S, number of L points in $S \approx \mathrm{vol}(S)/\det(L)$

Lattices: Basic Concepts

Why is the determinant $\det(L(B)) = |\det(B)|$ a property of the lattice L and not dependent on the particular basis B? Recall:

Lemma (Relation of lattice bases)

Any two bases B, B' of a given lattice L are related by $B' = B \cdot U$ for some matrix $U \in \mathbb{Z}^{n \times n}$ with $\det U \in \{-1, 1\}$.

As a consequence, any two bases of L have the same (absolute) determinant:

$$|\det(B')| = |\det(B \cdot U)| = |\det(B) \cdot \det(U)| = |\det(B)| \cdot |\det(U)| = |\det(B)|.$$

Hence, the determinant (FP volume) is a lattice property, invariant of the basis used.

Lattices: Basic Concepts

Sometimes, useful to remove from each basis vector its components along the previous basis vectors:

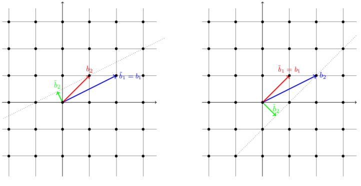
Definition

For a lattice basis $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$, its Gram-Schmidt Orthogonalization (GSO) is the matrix of vectors $B^* = (\vec{b}^*_1, \vec{b}^*_2, \ldots, \vec{b}^*_n)$ defined by $\vec{b}^*_1 = \vec{b}_1$ and for $i \geq 2$,

$$\vec{b}^*_i = \vec{b}_i - \sum_{j=1}^{i-1} \mu_{i,j} \cdot \vec{b}^*_j, \text{ where } \mu_{i,j} = \frac{\langle \vec{b}_i, \vec{b}^*_j \rangle}{\langle \vec{b}^*_j, \vec{b}^*_j \rangle}.$$

Example of GSOs in 2-Dimensions:

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \end{bmatrix}$$



Lattices: Basic Concepts

Can view GSO transformation as re-writing the coordinates of $\vec{b}_i$'s in a rotated coordinate system along $\vec{b}_i^*$s:

$$\begin{bmatrix} | & \cdots & | \\ \vec{b}_1 & \ddots & \vec{b}_n \\ | & \cdots & | \end{bmatrix} = \begin{bmatrix} | & \cdots & | \\ \vec{b}_1^* & \ddots & \vec{b}_n^* \\ | & \cdots & | \end{bmatrix} \cdot \begin{bmatrix} 1 & \mu_{2,1} & \cdots & \mu_{n,1} \\ 0 & 1 & \cdots & \mu_{n,2} \\ 0 & 0 & \cdots & \mu_{n,3} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

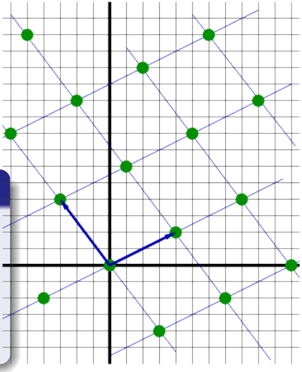
$$= \begin{bmatrix} | & \cdots & | \\ \frac{\vec{b}_1^*}{\|\vec{b}_1^*\|} & \ddots & \frac{\vec{b}_n^*}{\|\vec{b}_n^*\|} \\ | & \cdots & | \end{bmatrix} \cdot \begin{bmatrix} \|\vec{b}_1^*\| & \|\vec{b}_1^*\| \cdot \mu_{2,1} & \cdots & \|\vec{b}_1^*\| \cdot \mu_{n,1} \\ 0 & \|\vec{b}_2^*\| & \cdots & \|\vec{b}_2^*\| \cdot \mu_{n,2} \\ 0 & 0 & \cdots & \|\vec{b}_3^*\| \cdot \mu_{n,3} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \|\vec{b}_n^*\| \end{bmatrix}$$

- ith column of Bottom RHS matrix = coordinates of $\vec{b}_i$ in the rotated coordinate system
- From last row, every non-zero lattice vector has length $\geq \|\vec{b}_n^*\|$.
- Because $\vec{b}_i^*$'s are orthogonal, the FP of B^* is a n-dimensional cube of side lengths $\|\vec{b}_i^*\|$:
  $\det L(B) = |\det(B)| = |\det(B^*)| = \prod_{i=1}^n \|\vec{b}_i^*\|$.

Lattices Background: Shortest Vector Problem (SVP)

For crypto. security, need computationally hard lattice problems.
Many problems related to geometry of lattices seem to be hard!

The most basic geometric quantity about a lattice is its minimum (aka Minkowski first minimum).



Definition

For an n-dim. lattice L, its minimum $\lambda(L)$ is the length of the shortest non-zero vector of L:
$\lambda(L) = \min(\|\vec{b}\| : \vec{b} \in L \setminus \mathbf{0})$

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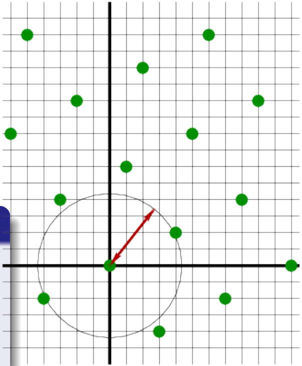

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Lattices Background: Minkowski's Theorem

For a given lattice L, how large can the lattice minimum $\lambda(L)$ be?

Theorem (Minkowski's First Theorem)

For any n-dim. lattice L, we have $\lambda(L) \leq \sqrt{n} \cdot \det L^{1/n}$.

Proof Idea: An analogue of the Pigeon-hole principle.

Lattices Background: Shortest Vector Problem (SVP)

Finding a vector of approximately minimum length seems to be hard, as the dimension n grows.

γ-Shortest Vector Problem (γ-SVP)

Given basis B for n-dim. lattice, find $\vec{b} \in L$ with: $0 < \|\vec{b}\| \leq \gamma \cdot \lambda(L)$.

Hardness of γ-SVP increases as approximation factor γ decreases:

- For $\gamma \geq 2^{O(n)}$: Easy – LLL algorithm solves in $\mathcal{P}oly(n)$ time.
- For $\gamma \leq O(1)$: NP-Hard (under randomized reductions) – very unlikely $\mathcal{P}oly(n)$ time algorithm exists.
- For crypto, need $\gamma = O(n^c)$ for some constant $c \geq 1/2$:
  - Best known attack algorithm time $T = 2^{O(n)}$ (even 'quantumly'!)
  - Best known γ-Time tradeoff: $T = \min(2^{O(n)}, 2^{O(n \log n)/\log \gamma})$.
  - Seems harder than Integer Factorization and Discrete Log.

Lattices Background: Cryptographic Lattices – q-ary lattices and SVP

Hardness of γ-SVP problem instance strongly depends on the given lattice basis B:

- There are many easy instances of γ-SVP, even for $\gamma = 1$ ('NP hard' case). Simple example: $B = I$.

In crypto., need to generate random lattices bases for which γ-SVP is hard to solve 'on average'.

- How to generate such 'hard' random lattices?

One possible answer (Ajtai, 1996): Generate a random q-ary lattice!

Lattices Background: Cryptographic q-ary lattices and SIS Problem

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One possible answer (Ajtai '96): a random q-ary lattice!

Ajtai's Random q-ary 'perp' lattices

Given an integer q and a uniformly random matrix $A \in \mathbb{Z}_q^{n \times m}$, the q-ary perp lattice $L_q^{\perp}(A)$ is defined by:

$$L_q^{\perp}(A) = \{\vec{v} \in \mathbb{Z}^m : A \cdot \vec{v} = \vec{0} \bmod q\}.$$

Lattices Background: Cryptographic q-ary lattices and SIS Problem

- γ-SVP problem for random q-ary perp lattices seems to be hard on average
  - Ajtai proved it, assuming γ-SVP is hard in the worst-case – see end of this module!
- Hardness of this computational problem is security basis for most of lattice-based cryptography.
- Known in lattice-based cryptography as the Small Integer Solution (SIS) Problem.

Problem

Small Integer Solution Problem – $SIS_{q,m,n,\beta}$: *Given n and a matrix A sampled uniformly in $\mathbb{Z}_q^{n\times m}$, find $\vec{v} \in \mathbb{Z}^m \setminus \{\vec{0}\}$ such that $A\vec{v} = \vec{0} \mod q$ and $\|\vec{v}\| \leq \beta$.*

Relation between SIS and γ-SVP

Problem

Small Integer Solution Problem – *$SIS_{q,m,n,\beta}$: Given n and a matrix A sampled uniformly in $\mathbb{Z}_q^{n \times m}$, find $\vec{v} \in \mathbb{Z}^m \setminus \{\vec{0}\}$ such that $A\vec{v} = \vec{0} \mod q$ and $\|\vec{v}\| \leq \beta$.*

Explicit relation of to γ-SVP:

- We have $\det(L_q^{\perp}(A)) = q^n$ (see week 2 tutorial).
- By Minkowski's Theorem, $\lambda(L_q^{\perp}(A)) \leq \sqrt{m} q^{n/m} \approx \sqrt{m}$ for $m \geq n \log q$.
- If Minkowski bound is good, then $SIS_{q,m,\beta} = \gamma$-SVP for $L_q^{\perp}(A)$, with $\gamma \approx \beta / \sqrt{m} q^{n/m}$ (practical refinement to Minkowski bound to be discussed next week).

Crypto. Application: Ajtai's Cryptographic Hash Function

How to use the hardness of SIS problem in cryptography?
First application: Collision-Resistant Hash Function (CRHF).

Definition

Ajtai's Hash Function $g_{q,m,n,d,A}$: Pick $A = (a_{i,j})$ uniformly random $n \times m$ matrix over $\mathbb{Z}_q$ (A = function 'public key'). Given input $\vec{x} \in \mathbb{Z}^m$ having 'small' coordinates ($\|\vec{x}\|_\infty \leq d$), hash function output is defined as

$$g_{q,m,n,d,A}(\vec{x}) = A \cdot \vec{x} \bmod q.$$

$$g(\vec{x}) = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & \cdots & a_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & \cdots & a_{n,m} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \\ x_m \end{bmatrix} \bmod q$$

Collision Resistance Security from SIS Problem

- Choose parameters such that domain is larger than range – collisions for f exist: $(2d+1)^m > q^n$.
- e.g., for compression ratio 2, may have $d = 1$, $m = 2 \cdot n \log q / \log(3)$.

Q: Why is it collision-resistant, assuming that SIS is a hard problem?

A: Collision-Resistance Security Reduction from SIS

- We show how to build an efficient SIS algorithm S, given an efficient collision-finder algorithm CF for function g.

Collision Resistance Security from SIS Problem

Suppose there was an efficient collision-finder attack algorithm CF for function g:

- Given random key (A, q) for function g_A, CF runs in time T_B and outputs a collision pair $\vec{x}_1 \neq \vec{x}_2$.

Then, given a SIS instance (A, q), SIS algorithm S:

- Runs collision-finder CF on input (A, q). CF outputs $\vec{x}_1 \neq \vec{x}_2$.
- S outputs SIS problem solution $\vec{v} = \vec{x}_1 - \vec{x}_2$.

Why does S work?

- A collision $\vec{x}_1 \neq \vec{x}_2$ gives a 'short' non-zero vector in $L_q^{\perp}(A)$:

  $$A\vec{x}_1 = A\vec{x}_2 \bmod q \Rightarrow \vec{v} = \vec{x}_1 - \vec{x}_2 \in L_q^{\perp}(A) \setminus \{\vec{0}\}, \|\vec{v}\| \leq \beta,$$

  where $\beta = 2\sqrt{m} \cdot d$.
- S is efficient (run-time $T_S \approx T_{\mathrm{CF}}$) if CF is efficient.

We proved **Theorem:** Collision-Resistance of g is (at least) as hard as $SIS_{q,m,n,\beta}$ with $\beta = 2\sqrt{m} \cdot d$.

Security of Lattice-Based Cryptography

- **Q1:** How should we choose the parameters q, m, n, d of Ajtai's hash function?
- **Q2:** How hard (secure) is SIS Problem and related γ-SVP problem?

Next week: We attempt to answer these questions.